Linear Algebra I

30/10/2026, Friday, 15:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Null space

$8 + 12 = 20$ pts

Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 1 & 3 \\ 3 & 0 & 3 & 3 \end{bmatrix}.$$

(a) Compute the reduced row echelon form of A. Hence determine the rank of A.

(b) Find a basis for the null space $N(A)$.

2 Characteristic polynomial

$2 + 1 + 1 + 2 + 1 + 3 + 3 + 2 + 3 + 3 + 4 = 25$ pts

Let $M \in \mathbb{R}^{4\times 4}$ have characteristic polynomial $p_M(\lambda) = (\lambda^2 - a^2)(\lambda^2 + a^2)$, where $a \in \mathbb{R}$. Justify all your answers.

(a) Determine the eigenvalues of M.

(b) Determine the determinant of M.

(c) Determine the trace of M.

(d) For which values of a is M nonsingular?

(e) What is the rank of M when $a \neq 0$?

(f) What can you say about the rank of M when $a = 0$?

(g) Is M diagonalizable when $a \neq 0$?

(h) Suppose that $a = 0$. Give an example of M that is not diagonalizable.

(i) Can M be symmetric?

(j) Can M be skew-symmetric ($M^T = -M$)?

(k) Show that $M^{30102026} = a^{30102024} M^2$.

3 Polynomials of matrices

20 pts

Let $A \in \mathbb{C}^{n\times n}$ and let $p(\lambda) = \lambda^k + p_{k-1}\lambda^{k-1} + \cdots + p_0$ be a polynomial with $k \geqslant 1$. Show that $p(A)$ is nonsingular if and only if none of the roots of p is an eigenvalue of A.

4 Unitary diagonalization

$5 + (10 + 10) = 25$ pts

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$ be linearly independent vectors. Define $M \in \mathbb{R}^{3\times 3}$ by

$$M = \boldsymbol{x}\boldsymbol{x}^T + \boldsymbol{y}\boldsymbol{y}^T + \boldsymbol{z}\boldsymbol{z}^T.$$

(a) Is M unitarily diagonalizable? Justify your answer.

(b) Suppose, in addition, that

$$\boldsymbol{x}^T\boldsymbol{x} = \boldsymbol{y}^T\boldsymbol{y} = \boldsymbol{z}^T\boldsymbol{z} = 2, \qquad \boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{z} = \boldsymbol{y}^T\boldsymbol{z} = 1.$$

(i) Find all eigenvalues of M and corresponding eigenvectors.
HINT: Explain why every eigenvector has the form $\boldsymbol{v} = a\boldsymbol{x} + b\boldsymbol{y} + c\boldsymbol{z}$ and determine the eigenvalues and eigenvectors from $M\boldsymbol{v} = \lambda\boldsymbol{v}$.

(ii) Find a unitary diagonalizer for M.

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10 pts free